DESENSITIZED OPTIMAL TRAJECTORIES WITH CONTROL CONSTRAINTS*

Hans Seywald

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Analytical Mechanics Associates, Inc.
303 Butler Farm Rd., Suite 104 A
Hampton, VA 23666
web: www.ama-inc.com
Phone: (757) 865-0944
email: seywald@ama-inc.com

Abstract

The concept of Desensitized Optimal Control (DOC) provides a framework to calculate trajectories that optimally balance the performance in the nominal case as well as in case when state perturbations are encountered at any time along the trajectory. In the present paper, the DOC concept is extended to problems with control constraints. The approach is illustrated on a vertical rocket landing problem.

1. Introduction

The basic idea behind the Desensitized Optimal Control (DOC) approach introduced in [1], [2] is to consider, in parallel, along with the optimal reference solution, a field of neighboring solutions onto which the actual trajectory can jump in case state perturbations are encountered. Numerically, the neighboring extremal field is approximately characterized through the variational equations about the nominal trajectory. The cost criterion with respect to which the reference solution and the embedding neighboring trajectory field are optimized is a user-defined function of i) physical quantities associated with the reference trajectory (such as final states, final time, etc.), and ii) the rate at which the values of these physical quantities change as the actual trajectory shifts from its nominal path to a neighboring path when state perturbations are encountered.

In [2], the DOC approach was introduced for the most basic optimal control problem without state and/or control constraints. The present paper extends the basic concept of Desensitized Optimal Control (DOC) to optimal control problems with control constraints. The underlying idea is to reduce the rate at which the control can change from the reference solution to a neighboring solution whenever the control associated with the reference solution approaches one of its bounds. Loosely speaking, this approach is analogous to reducing the controller gains gradually to zero whenever the reference control approaches one of its bounds. In the DOC approach it is left up to the optimizer to balance this loss of controllability with any gains that can be obtained from operating the reference controls near saturation.

It is not the goal of this paper to provide a thorough mathematical derivation and/or justification of the pertinent ideas underlying the here proposed extension of the DOC approach to control constrained problems. Instead, the intention is to simply convey to the reader the underlying ideas in the most ilustrative form possible. Explicitly, this is attempted by treating a practical example, namely a vertical rocket landing problem with control constraints.

2. Vertical Rocket Landing Problem

We consider the following optimal control problem:

$$\min_{u \in (PWC[t_0, t_f]), t_f \in \mathbf{R}} m(t_f) \tag{1}$$

subject to the equations of motion

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= T_{max} \cdot u - g \\ \dot{m} &= \frac{T_{max}}{v_e} \cdot u \end{aligned} \tag{2}$$

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the control constraint

$$\begin{aligned} u &\geq u_{min} \\ u &\leq u_{max} \end{aligned} \tag{3}$$

the initial conditions

$$\begin{aligned} x(0) &= 10 \\ v(0) &= 0 \\ m(0) &= 1 \end{aligned} \tag{4}$$

and the final condition

$$\begin{aligned} x(t_f) &= 0.1 \\ v(t_f) &= 0 \end{aligned} \tag{5}$$

Here, x denotes the distance from the ground, v denotes the vertical velocity, and m denotes the fuel consumed. The constants g, v_e, u_{min}, u_{max}, and T_{max} represent the gravitational acceleration, the exhaust velocity, the control bounds, and the maximum available thrust, respectively. For numerical calculations, we use the following values:

$$\begin{aligned} g &= 1, \\ v_e &= 1, \\ u_{min} &= 0, \\ u_{max} &= 1, \\ T_{max} &= 3. \end{aligned} \tag{6}$$

It is well-known that the optimal solution to problem (1)-(6) is of bang-bang nature. Explicitly, the fuel-optimal strategy is to use minimum thrust as long as possible, and switch over to maximum thrust at the latest possible time such that touch-down is achieved with zero relative velcoity to the ground. Obviously, this strategy is not recommended in the presense of model uncertainties as it leaves no degree of freedom for corrections once the maximum thrust arc is reached.

In the following, we want to use the Desensitized Optimal Control approach to find solutions that balance the desire for efficiency (fuel optimality) and the need for safety margins in an optimal way.

3. Desensitized Optimal Control (DOC) Formulation

For the sake of simplicity, we want to consider as the sole source for uncertainties the possibility that the actual value of T_{max} may be slightly different from its nominal value. Our task is then to redesign the descent trajectory such that the nominal fuel consumption is still reasonably good, while the sensitivity of the degree to which the final conditions (5) are violated as a result of perturbations in T_{max} is kept at a reasonably low level. To achieve this we formally apply the DOC approach introduced in [2] to problem (1)-(6). It should be noted, however, that the approach in [2] is applicable only to problems without state and control constraints. Note that conditions (3) constitute control constraints.

The approach in the present section is to first state the conditions that follow from formally applying the standard DOC approach of [2] to problem (1)-(6). Then we describe the nature of the difficulties caused by the control constraints (3). A method to overcome these difficulties is presented in the next section.

The physical goal is to achieve reasonably low fuel consumption and at the same time reduce the sensitivity of the final values of x and v with respect to perturbations in the maximum availble thrust T_{max}. Following the standard DOC approach introduced in [2] this leads to the problem of finding the control function of time $u \in (PWC[t_0, t_f])$ and the final time t_f such that the performance index

$$J = m(t_f) + \alpha \cdot \int_0^{t_f} P_{1,4}^2(t) + P_{2,4}^2(t)\, dt \tag{7}$$

where

$$P(t) = S(t_f) \cdot S(t)^{-1} \tag{8}$$

is minimized subject to the equations of motion

$$\begin{aligned} \dot{x} &= v && (9)\\ \dot{v} &= T_{max} \cdot u - g && (10)\\ \dot{m} &= \frac{T_{max}}{v_e} \cdot u && (11)\\ \dot{T}_{max} &= 0 && (12)\\ \dot{S} &= \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \cdot K\right) \cdot S && (13) \end{aligned}$$

the control constraints

$$\begin{aligned} u &\geq 0 \\ u &\leq 1 \end{aligned} \qquad (14)$$

the initial conditions

$$\begin{aligned} x(0) &= 10 \\ v(0) &= 0 \\ m(0) &= 1 \\ T_{max}(0) &= 3 \\ S(0) &= I_{4\times4} \end{aligned} \qquad (15)$$

and the final conditions

$$\begin{aligned} x(t_f) &= 0.1 \\ v(t_f) &= 0 \end{aligned} \qquad (16)$$

In (7), $P_{ij}^2(t)$ denotes the square of the ij-element of the matrix function $P(t)$ defined in (8). Equations (9), (10), (11) are identical to the original equations of motion (2). Equation (12) is introduced to raise the status of the fixed parameter T_{max} to that of a state. This approach was suggested in [2] to enable the application of the DOC approach to parameter uncertainties. The 4×4-matrix function $S(t)$ obtained from solving the initial value problem (13), (15) represents the sensitivity of the state vector $y = [x, v, m, T_{max}]^T$ at time t with respect to perturbations in y at the initial time, zero. Following [2], the matrix function $P(t)$ defined in (8) then represents the sensitivity of the state vector y at final time t_f with respect to perturbations in the state y at the "current" time t. The symbol f in (13) represents the four-dimensional vector of right-hand sides in the $\dot{y}$-equation, i.e. the right-hand sides of the equations of motion (9)-(12). The symbol K in (13) denotes the control gain $K(t) = \frac{\partial u(t)}{\partial x(t)}$. Thus, along with the reference control $u^*(t)$ the row vector $K(t) = [K_1(t), K_2(t), K_3(t), K_4(t)]$ defines the control actions in the neighborhood of the optimal reference solution through the linear feedback law (see [2])

$$u(t) = u^*(t) + K(t) \cdot (x(t) - x^*(t)), \qquad (17)$$

Here superscript * denotes quantities associated with the optimal solution to problem (7)-(16). Following [2], the gains K_i can be either prescribed a priori by the user, or can be treated like additional control-like variables that are optimized along with the reference control u^*.

Now recall that the DOC approach outlined in [2] is not applicable to problems with control constraints and/or state constraints. Obviously, in the presence of control constraints such as (14), the feedback law (17) may lead to inadmissible control actions. In fact, along arcs where a control bound is satisfied with strict equality an arbitrarily small perturbation in the physical state vector y could yield an inadmissible control action unless all gains are equal zero.

In the next section we introduce a simple method to avoid this difficulty.

4. Treatment of Control Constraints

In the previous section it was observed that, along control constrained arcs, the feedback law (17) is prone to lead to inadmissible control commands. Clearly, this observation can be generailized to the following statement: the closer the optimal reference control is to any of its prescribed bounds, the smaller the gain vector K needs to be chosen to prevent the feedback law (17) from yielding inadmissible control commands. An engineering approach to resolve this difficulty is hence to multiply the gain vector K with a factor that shrinks to zero whenever the reference control appraoaches one of its bounds, e.g.

$$K = \hat{K} \cdot \zeta(u), \tag{18}$$

with

$$\zeta(u) = \frac{(u - u_{min}) \cdot (u_{max} - u)}{\left(\frac{u_{max} - u_{min}}{2}\right)^2}. \tag{19}$$

In (18), $\hat{K(t)} = [\hat{K}_1(t), \hat{K}_2(t), \hat{K}_3(t), \hat{K}_4(t)]$ is a row vector of gain functions of time that may either be prescribed a priori by the user, or optimized along with u.

By introducing the factor $\zeta(u)$ into the gain vector K it becomes the task of the overall optimal control problem (7)-(16), (18), (19) to balance between moving the control closer to the bound to improve the nominal cost or to chose intermediate control to reduce sensitivity.

5. Numerical Results

The numerical task is to solve the two-point boundary value problem (7)-(16), (18), (19). For the sake of simplicity, the gain functions $\hat{K}(t) = [\hat{K}_1(t), \hat{K}_2(t), \hat{K}_3(t), \hat{K}_4(t)]$ are set constant, rather than optimizing them along with the reference control $u^*(t)$. Explicitly, we set

$$\hat{K}(t) = [-1, -1, -1, -1]. \tag{20}$$

The choice of signs in (20) reflects the logic that the control u obtained from (17) needs to be decreased whenever any of the states x, v, T_{max} or m is individually perturbed to a larger than nominal value at any time along the nominal solution. The penalty factor $\alpha \geq 0$ in (7) is varied between 0 and 10. For $\alpha = 0$, no sensitivity penalty is applied at all. In this case, the components of the sensitivity matrix S are integrated along merely as dummy variables without having any effect on the optimal solution. As α is increased, the importance of the sensitivity of the states $x(t_f)$ and $v(t_f)$ with respect to perturbations in $T_{max}(t)$ is steadily increased in the cost function (7).

The resulting optimal control problems (7)-(16), (18), (19) are solved approximately using a collocation-type direct optimization approach [3] with 101 equidistantly placed nodes. Time optimal control histories for the cases $\alpha = 0$, 0.1, 1, and 10 are shown in Figure 1a. Note that, as α is increased, the optimal reference control stays away further from its maximum value, 1, effectively leaving room for control corrections, should state perturbations make them necessary. Interestingly, for all values of α, the control is identically zero during the initial part of the time interval. Figure 1b shows the percentage change in fuel consumption as a function of the penalty factor α.

The difference between regular optimal control solutions and desensitized optimal control solutions is best demonstrated in simulations. Figure 2 shows the dispersion in the final states $x(t_f)$ and $v(t_f)$ obtained from perturbing the value of T_{max} randomly between $\pm 5\%$ of its nominal value, 3, and using the optimal reference controls of Figure 1a along with the feedback law determined by (17), (18), (19). From Figure 1b it can be seen that the reduction in sensitivity comes at the cost of only a modest increase in fuel consumption.

6. Summary

The Desensitized Optimal Control (DOC) approach introduced in [1], [2] is extended to control constrained optimal control problems. The general idea behind this step is demonstrated on a vertical rocket landing problem. The benefit of the DOC solutions over regular optimal control solutions is demonstrated through simulations. As expected by virtue of their design, DOC solutions lead to smaller dispersions of the final states as a result of state or parameter perturbations encountered over the course of the trajectory. For the example of the vertical rocket landing problem treated in this paper, the reduction in sensitivity comes at the cost of only a modest increase in fuel consumption.

References

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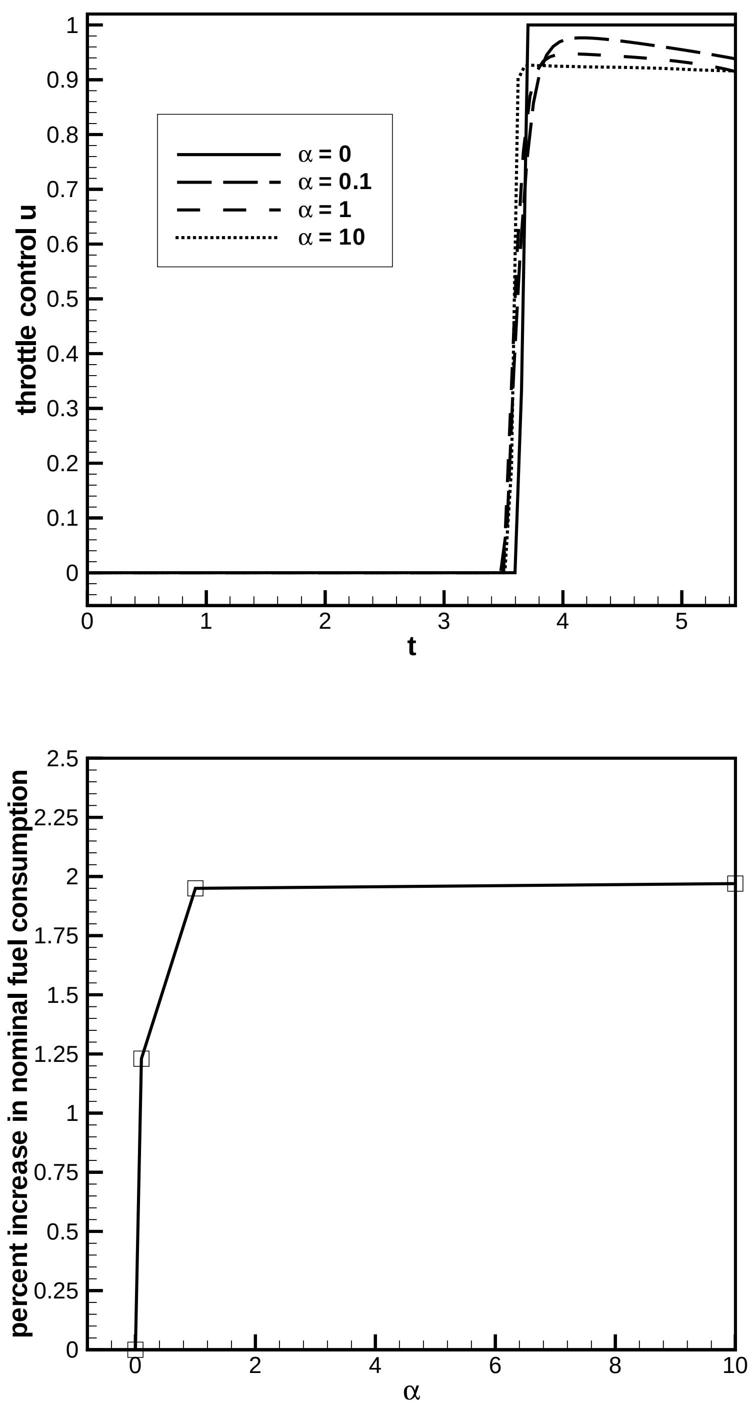

Figure 1: Time histories of reference control and associated fuel consumption obtained for various penalty factors α

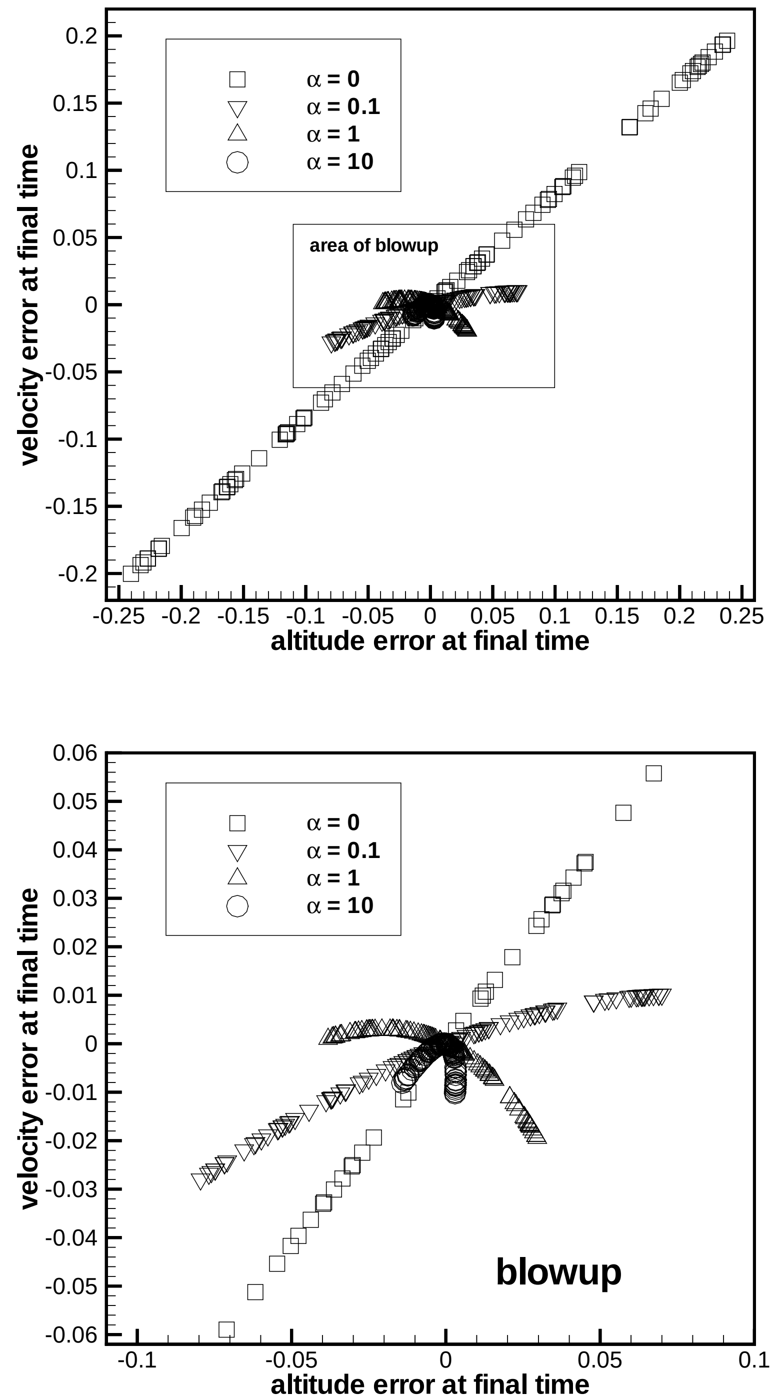

Figure 2: Dispersion of final states obtained in simulation runs with the maximum thrust perturbed randomly between ±5% of its nominal value, 3